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Zohra Kader *,** **Christophe Fiter** * **Laurentiu Hetel** ***
Lotfi Belkoura *,**

* *Centre de Recherche en Informatique, Signal et Automatique de Lille (CRIStAL), UMR CNRS 9189, Université Lille 1 Science et Technologies, 59650 Villeneuve d'Ascq, France.*
** *Non-A, INRIA - Lille Nord Europe, 40 avenue Halley, Villeneuve d'Ascq, 59650, France.*
*** *Centre de Recherche en Informatique, Signal et Automatique de Lille (CRIStAL), UMR CNRS 9189, Ecole Centrale de Lille, 59650 Villeneuve d'Ascq, France.(e-mails: zohra.kader@inria.fr,christophe.fiter@univ-lille1.fr, laurentiu.hetel@ec-lille.fr, lotfi.belkoura@univ-lille1.fr)*

Abstract: In this paper we consider the problem of non-quadratic stabilization of switched affine systems. Using a switching Lyaponuv function, LMI conditions based constructive method are proposed in order to design nonlinear switching surfaces and provide an estimation of the domain of attraction. An illustrative example is given in order to show the efficiency of the proposed method.

Keywords: Hybrid dynamical systems, Switched systems, nonlinear switching surfaces, bilinear systems, Lur'e Lyapunov function, local stabilization.

1. INTRODUCTION

Switched affine systems present a simple class of hybrid systems [Antsaklis 1998], [Goebel et al. 2012], [Liberzon 2003]. They are widely used in different application fields [Deaecto et al. 2010], [Defoort et al. 2011].

They may present complex phenomena which make their study very challenging : zeno solutions, limit cycles, chattering or sliding modes (see for instance the particular case of relay systems in [Gonçalves et al. 2001], [Johansson et al. 2002], [Johansson et al. 1999], and [Utkin et al. 2009]). Various approaches have been proposed for stabilizing state-dependent switching law design in the literature see for instance [Deaecto et al. 2010], [Hetel and Bernuau 2015], [Riedinger et al. 2010], and [Trofino et al. 2012]. However, the problem of designing state-dependent switching laws for switched affine systems is still widely open. Recently, a convex embedding formalism has been used in order to design a stabilizing switching law by [Hetel and Bernuau 2015]. The design procedure consists in the use of the existence of an exponentially globally stabilizing state feedback controller as a reference control to be emulated by a switching controller. The designed switching surfaces are linear and local stabilization of the closed loop-system is ensured in an ellipsoidal domain of attraction obtained by the use of a quadratic Lyaponuv function. The work in this paper is in the same spirit. However, we extend the approach in several directions. Here, we propose a numerical method for the design of nonlinear switching surfaces using switched Lyapunov function. An estimation of the non-ellipsoidal domain of attraction is also provided. The result can be related to the simplex method [Bartolini et al. 2011]. We use numerical tools which are inspired from convex optimization approaches employed for systems with saturated actuators [Blanchini 1999], [Boyd et al. 1994], [Tarbouriech et al. 2011]. In order to illustrate the efficiency of the proposed methods a numerical example is given.

The paper is structured as follows: Section 2 gives some preliminaries and exposes the problem under study. In Section 3 we propose an LMI based approach allowing the design of nonlinear state-dependent switching laws that ensure the stability of switched affine systems and provide a non-ellipsoidal estimation of the domain of attraction. In Section 4, a numerical example that illustrates the efficiency of the proposed method is provided. The paper is ended with a brief conclusion.

1.1 Notations

The transpose of a matrix M is denoted by M^T and if the matrix is symmetric the symmetric elements are denoted by $*$. The notation $M \succeq 0$ (resp. $M \preceq 0$) means that the matrix M is positive (resp. negative) semi-definite, and the notation $M \succ 0$ (resp. $M \prec 0$) means that it is positive definite (resp. negative definite). The identity matrix is denoted by I. The notation $M_{(i)}$ refer to the i-th row of a matrix (or vector) M.

For a positive integer N, we denote by $\mathcal{I}_N$ the set $\{1, \ldots, N\}$. By Δ_N we denote the unit simplex

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$$\Delta_N = \left\{ \beta = \begin{bmatrix} \beta_{(1)} \\ \vdots \\ \beta_{(N)} \end{bmatrix}^T \in \mathbb{R}^N : \sum_{i=1}^{N} \beta_{(i)} = 1, \beta_{(i)} \geq 0, i \in \mathcal{I}_N \right\}. \tag{1}$$

A vector $v \in \mathbb{R}^m$ is said to be strictly positive if for all $i \in \mathcal{I}_m$, $v_{(i)} > 0$.

For a positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a positive scalar γ, we denote by $\mathcal{E}(P, \gamma)$ the ellipsoid

$$\mathcal{E}(P, \gamma) = \{x \in \mathbb{R}^n : x^T P x \leq \gamma\}, \tag{2}$$

and for all positive scalar r, we denote by $\mathcal{B}(0, r)$ the ball

$$\mathcal{B}(0, r) = \mathcal{E}(I, r) = \{x \in \mathbb{R}^n : x^T x \leq r\}. \tag{3}$$

For a given set $\mathcal{S}$, the notation $\mathrm{Conv}\{\mathcal{S}\}$ indicates the convex hull of the set, $\mathrm{int}\{\mathcal{S}\}$ its interior and $\overline{\mathcal{S}}$ its closure. The closed convex hull of the set $\mathcal{S}$ will be noted by $\overline{\mathrm{Conv}\{\mathcal{S}\}}$. Finally, we denote by $\mathrm{Vert}\{\mathcal{S}\}$ the set of vertices of $\mathcal{S}$. Let $\mathcal{S} \subset \mathbb{R}^m$ be a finite set of vectors. The minimum argument of a given function $f : \mathcal{S} \longrightarrow \mathbb{R}$ is noted by

$$\arg\min_{x \in \mathcal{S}} f(x) = \{y \in \mathcal{S} : f(y) \leq f(z), \forall z \in \mathcal{S}\}.$$

2. PRELIMINARIES

2.1 System description

Consider the system

$$\dot{x} = Ax + \sum_{k=1}^{m} (N_k x + b_k) u_{(k)}, \tag{4}$$

with $x \in \mathbb{R}^n$ and $u_{(k)}$ the k-th component of the input u. The input u is only allowed to take values in the set $\mathcal{V} = \{v_1, \ldots, v_N\} \subset \mathbb{R}^m$. $A \in \mathbb{R}^{n \times n}$, $b_k \in \mathbb{R}^{n \times 1}$ and $N_k \in \mathbb{R}^{n \times n}$ are the matrices describing the system. We use the notation $B = [b_1, \ldots, b_m]$.

It has been demonstrated in [Hetel and Bernuau 2015] that the class of switched system (4) is quite general in the sense that any switched affine system

$$\begin{aligned} \dot{x} &= \tilde{A}_\kappa x + \tilde{b}_\kappa, \\ \kappa &\in \mathcal{I}_N, \end{aligned} \tag{5}$$

with $\tilde{A}_i \in \mathbb{R}^{n \times n}$ and $\tilde{b}_i \in \mathbb{R}^{n \times 1}$ the matrices describing the subsystems, can be represented in the form (4) (see Proposition 1 in [Hetel and Bernuau 2015]). In the sequel we assume that:

- A-1 The pairs $(A(v_i), B)$, for all $i \in \mathcal{I}_N$ with $A(v_i) = A + \sum_{k=1}^{m} N_k v_{i(k)}$ are simultaneously quadratically stabilizable. This means that there exist matrices K, $P = P^T \succ 0$ such that
$$A_{cl}(v_i)^T P + P A_{cl}(v_i) \prec 0, \forall i \in \mathcal{I}_N$$
with $A_{cl}(v_i) = A(v_i) + BK$.
- A-2 The set $\mathrm{int}\{\mathrm{Conv}\{\mathcal{V}\}\}$ is nonempty and the null vector is contained inside ($0 \in \mathrm{int}\{\mathrm{Conv}\{\mathcal{V}\}\}$).

Note that for any finite set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ there exists a finite number n_l of vectors $l_i \in \mathbb{R}^{1 \times m}$, $i \in \mathcal{I}_{n_l}$ such that

$$\mathrm{Conv}\{\mathcal{V}\} = \{u \in \mathbb{R}^m : l_i u \leq 1, \forall i \in \mathcal{I}_{n_l}\}. \tag{6}$$

Typical control sets $\mathcal{V}$ for switched affine systems are often of the form

$$\mathcal{V} = \mathrm{Vert}\{\mathcal{P}(c)\}, \tag{7}$$

where the hyperrectangle $\mathcal{P}(c)$, with c a strictly positive vector, is given by

$$\mathcal{P}(c) = \left\{ u = \begin{bmatrix} u_{(1)} \\ \vdots \\ u_{(m)} \end{bmatrix}^T \in \mathbb{R}^m : \left| u_{(k)} \right| \leq c_{(k)}, \forall k \in \mathcal{I}_m \right\}. \tag{8}$$

However, we want to keep the problem formulation as general as possible. Note that since A.2 holds, even for more general sets $\mathcal{V}$ there exists a vector $c \in \mathbb{R}^m$ such that the hyperrectangle satisfies $\mathcal{P}(c) \subseteq \mathrm{Conv}\{\mathcal{V}\}$. In the sequel, we will consider such a vector c and use the notation (8) to state the main results.

This paper deals with the stabilization of system (4). We consider a controller given by

$$u(x) \in \arg\min_{v \in \mathcal{V}} \Gamma(x, v), \tag{9}$$

where the mapping $\Gamma : \mathbb{R}^n \times \mathcal{V} \longrightarrow \mathbb{R}$ characterizes the switching surfaces.

The interconnection (4), (9) can be rewritten as

$$\dot{x} = Ax + \sum_{k=1}^{m} N_k x u_{(k)}(x) + Bu(x) = \mathcal{X}(x). \tag{10}$$

Note that this is a differential equation with a discontinuous right hand side $\mathcal{X}(x)$ [Cortes 2008], [Filippov and Arscott 1988]. Therefore, we consider the solutions in the sense of Filippov.

2.2 Recent results and problem statement

In this paper we are interested in the study of the following problem:

Problem 1. Given system (4) under Assumptions A-1 and A-2, the set $\mathcal{V}$ and a ball $\mathcal{B}(0, R)$, design a switching law (9) such that the closed loop system is locally asymptotically stable in a domain $\mathcal{D}$ such that $\mathcal{B}(0, R) \subseteq \mathcal{D}$.

In [Hetel and Bernuau 2015] a constructive numerical method for stabilizing state-dependent switching laws design is given. Assuming A-2 and

- A-1' There exist a positive definite matrix Q and positive scalars χ and α such that
$$A(v_i)Q + QA(v_i)^T - \chi BB^T \preceq -2\alpha Q, \forall i \in \mathcal{I}_N, \tag{11}$$

it is proved that system (4) with a switching law (9) is locally exponentially stable with a decay rate α. In that paper, a linear switching function is considered:

$$\Gamma(x, v) = -\frac{2}{\chi} x^T H^T v \tag{12}$$

with $H = -\frac{\chi}{2} B^T Q^{-1}$. Note that A-1' is equivalent to A-1 (this is similar to what is presented in Boyd et al. [1994] page 112). An estimation of the domain of attraction is equally given using a quadratic Lyapunov function $V(x) = x^T P x$ with $P = Q^{-1}$: $\mathcal{E}(P, \gamma)$ where γ is computed such that $\mathcal{E}(P, \gamma)$ does not cross the convex hull

$$\mathcal{C}_v(H) = \{x \in \mathbb{R}^n : l_i H x \leq 1, \forall i \in \mathcal{I}_{n_l}\}, \tag{13}$$

where $l_i, i \in \mathcal{I}_{n_l}$ are vectors defined in (6), which leads to $\gamma \leq \min_{i \in \mathcal{I}_{n_l}} (l_i H Q H^T l_i^T)^{-1}$. Nevertheless, considering a quadratic Lyapunov function, linear switching surfaces and an ellipsoidal estimation of the domain of attraction introduces some conservatism in the proposed method.

Here we would like to provide a more general design procedure using non-quadratic Lyapunov functions to compute nonlinear switching surfaces and non-ellipsoidal domains of attraction.

Using the above definitions, in the next section, we provide a constructive method based on LMI criteria for switched affine system (4). It allows the design of nonlinear stabilizing state-dependent switching surfaces and provides a larger estimation of the non-ellipsoidal domain of attraction.

3. STABILIZATION OF SWITCHED AFFINE SYSTEMS USING NONLINEAR SWITCHING SURFACES

In this section we provide numerical tools for nonlinear switching surfaces design.

Theorem 1. Consider system (4) and assume that A-1' (or equivalently A-1) and A-2 hold. If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, two diagonal positive definite matrices $\Omega \in \mathbb{R}^{m\times m}$ and $M \in \mathbb{R}^{m\times m}$, a matrix $\Upsilon \in \mathbb{R}^{m\times n}$, and a strictly positive vector $\tau \in \mathbb{R}^m$ such that for all $i \in \mathcal{I}_N$

$$\begin{bmatrix} A(v_i)_{cl}^T P + PA(v_i)_{cl} & PB - \Upsilon^T - A(v_i)_{cl}^T H^T \Omega \\ * & -2M - \Omega HB - (\Omega HB)^T \end{bmatrix} \prec 0, \tag{14}$$

and

$$\begin{bmatrix} P & M_{(k,k)} H_{(k)}^T - \Upsilon_{(k)}^T \\ M_{(k,k)} H_{(k)} - \Upsilon_{(k)} & \tau_{(k)} c_{(k)}^2 \end{bmatrix} \succeq 0, \forall k \in \mathcal{I}_m, \tag{15}$$

where $H = -\frac{\chi}{2} B^T Q^{-1}$ and $A(v_i)_{cl} = A + \sum_{k=1}^m N_k v_{i(k)} + BH$, then system (4) with the switching law

$$u \in \arg\min_{v\in\mathcal{V}} (x^T P - \phi(Hx)^T \Omega H) Bv \tag{16}$$

is locally asymptotically stable.

An estimation of the domain of attraction is given by

$$\mathcal{L}_V(r^{-1}) = \{x \in \mathbb{R}^n : V(x) \leq r^{-1}\}, \tag{17}$$

where V is defined as

$$V(x) = x^T Px - 2\sum_{k=1}^m \int_0^{H_{(k)}x} \phi_{(k)}(s)\Omega_{(k,k)} ds, \tag{18}$$

with $\phi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ a nonlinear function defined for all $y \in \mathbb{R}^m$ as $\phi(y) = \left[\phi_{(1)}(y_{(1)}), \ldots, \phi_{(m)}(y_{(m)})\right]^T \in \mathbb{R}^m$, with

$$\phi_{(k)}(\sigma) = \begin{cases} c_{(k)} - \sigma & \text{if} \quad \sigma > c_{(k)}, \\ 0 & \text{if} \ -c_{(k)} \leq \sigma \leq c_{(k)}, \ \forall\sigma \in \mathbb{R}. \\ -c_{(k)} - \sigma & \text{if} \quad \sigma < -c_{(k)}. \end{cases} \tag{19}$$

and $r \geq \max_{k\in\mathcal{I}_m} \left\{ \frac{\tau_{(k)}}{M_{(k,k)}^2} \right\} > 0.$

Proof. See the internal report [Kader et al. 2016]. ■

Remark 1. Assumption A-1' (respectively A-1) can be easily checked using LMI solvers.

4. ILLUSTRATIVE EXAMPLE

In order to illustrate the performance of the proposed control method, we consider the bilinear system (4) with matrices

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 4 & -5 \end{bmatrix}, \ B = \begin{bmatrix} 15 & 1 \\ -1 & -5 \end{bmatrix}, \\ N_1 &= \begin{bmatrix} 1 & -5 \\ 0.5 & 2 \end{bmatrix}, \ N_2 = \begin{bmatrix} -1 & 5 \\ -0.5 & -2 \end{bmatrix}, \end{aligned} \tag{20}$$

and the controller u which takes values in the set

$$\mathcal{V} = \left\{ \begin{bmatrix} 30 \\ 30 \end{bmatrix}, \begin{bmatrix} 30 \\ -30 \end{bmatrix}, \begin{bmatrix} -30 \\ 30 \end{bmatrix}, \begin{bmatrix} -30 \\ -30 \end{bmatrix} \right\}. \tag{21}$$

One can verify that the open-loop linear system is unstable (the eigenvalues of the matrix A are 2.12 and -6.12). Therefore, the proposed methods in the literature can not be used to stabilize the system. Choosing a decay rate $\alpha = 0.5$, we design the linear switching law proposed in [Hetel and Bernuau 2015] in order to stabilize the system to the origin. We obtain the following solutions of (11)

$$Q = \begin{bmatrix} 0.0629 & -0.0068 \\ -0.0068 & 0.0042 \end{bmatrix} \tag{22}$$

and $\chi = 0.1$.

We deduce then

$$H = \begin{bmatrix} 12.61 & -8.843 \\ 6.73 & 69.34 \end{bmatrix} \tag{23}$$

with an estimation of the ellipsoidal domain of attraction $\mathcal{E}(P, \gamma)$ where $\gamma = 54.1602$ and $P = Q^{-1}$. Based on Theorem 1, we design a nonlinear switching law by solving LMIs (14) and (15) for $P = Q^{-1}$ and H as given in (22)-(23) and a vector c satisfying (8) such that $c_{(1)} = c_{(2)} = 30$.

We obtain $\Omega = \begin{bmatrix} 1.075 & 0 \\ 0 & 0.34 \end{bmatrix}$ and $r^{-1} = 209.85$.

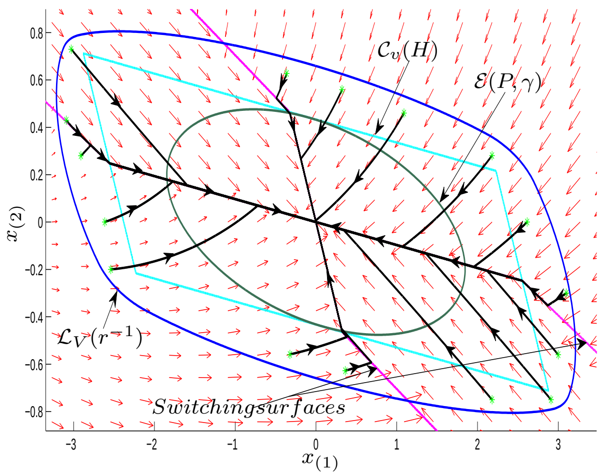

Fig. 1. Phase plot of system (4), (20), (21)

As we can see from Figure 1, the obtained trajectories starting in the domain of attraction $\mathcal{L}_V(r^{-1})$ converge to the origin. We can also note that the domain of attraction $\mathcal{L}_V(r^{-1})$ is larger than the ellipsoidal domain of attraction $\mathcal{E}(P, \gamma)$ obtained by the method proposed in [Hetel et al. 2013]. The nonlinear switching surfaces and the convex hull $\mathcal{C}_v(H)$ defined in (13), which limits the domain of attraction in the approach in [Hetel et al. 2013], are equally represented. The evolution of the state variables starting at $x(0) = [0.3227, 0.5590]^T$ are presented in Figure 2. We can observe that the state converges to zero.

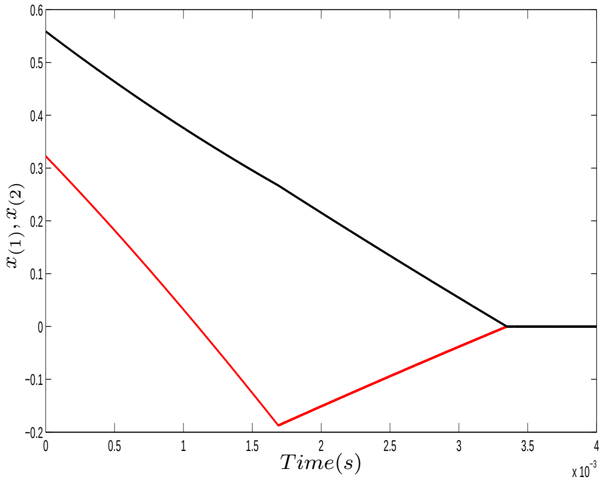

Fig. 2. Evolution of the state variables starting at $x(0) = [0.3227, 0.5590]^T$

5. CONCLUSION

This paper provides an approach for the design of stabilizing switching laws for switched affine systems. Non-quadratic Lyapunov functions are used to develop a method allowing the computation of nonlinear switching surfaces and the enlargement of the domain of attraction. LMI criteria are given in order to design the controller and provide an estimation of a non-ellipsoidal domain of attraction. In future works, we will consider robustness aspects with respect to perturbations, noises, etc.

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